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Two-Parameter Deformed SUSY Algebra for Fibonacci Oscillators

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Abstract We construct a two-parameter deformed SUSY algebra for the system of n ordinary fermions and $n(q_1, q_2)$-deformed bosons called Fibonacci oscillators with $SU_{q_1/q_2}(n)$-symmetry. We then analyze the Fock space representation of the algebra constructed. We obtain the total deformed Hamiltonian and the energy levels together with their degeneracies for the system. We also consider some physical applications of the Fibonacci oscillators with $SU_{q_1/q_2}(n)$-symmetry, and discuss the main reasons to consider two distinct deformation parameters.

Keywords Deformed bosons · Quantum groups · Quantum superalgebras

1 Introduction

The discovery of the new kind of mathematical structures called quantum groups and their associated algebras [34, 37, 51] have found many applications in several areas of research of theoretical physics such as noncommutative geometry [59, 75, 77], quantum mechanics [2, 57, 72, 73] and exactly solvable statistical models [33, 66]. Quantum groups or q-deformed Lie algebras imply some specific deformations of the usual Lie algebras. Mathematically speaking, they are defined algebraically as quasi-triangular Hopf algebras and they can be either noncommutative or commutative.

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Historically, the q-deformed boson algebra was constructed by Arik and Coon [9], Macfarlane [58], and Biedenharn [16]. Such developments have led to the discussions of q-deformed fermionic versions of the q-deformed bosonic oscillator algebras [48, 52, 74]. Hence, q-deformed oscillator constructions of supergroups and superalgebras by using these deformed boson and fermion oscillators have extensively been investigated [3, 6, 19–21, 25–27, 38, 49, 50, 56].

After the simplest $N = 2$ SUSY algebra was developed by Witten [76] for undeformed bosons and undeformed fermions, several q-deformed versions of the $N = 2$ SUSY algebra have been studied by using q-deformed boson and fermion algebras [4, 17, 23, 42, 47, 65, 71]. Different aspects of algebraic SUSY structures such as fractional supersymmetry and parasupersymmetry have been considered by using q-deformed bosons as well as fermions when the deformation parameter q is a root of unity [15, 22, 24, 30, 31, 35, 36, 53, 54, 61, 63].

In this paper, our aim is to introduce an alternative example of deformed SUSY algebra and construct a new two-parameter generalized SUSY algebra using (q_1, q_2)-deformed bosons called Fibonacci oscillators invariant under the quantum group $SU_r(n)$ with $r = q_1/q_2$ and undeformed fermions.

We begin this paper by reviewing the quantum group invariant two-parameter deformed boson algebra and discuss its representative properties. We then construct the SUSY quantum mechanics for $SU_{q_1/q_2}(n)$-invariant (q_1, q_2)-deformed bosons and undeformed fermions. This is followed by an analysis of the Fock space representation of the (q_1, q_2)-deformed SUSY algebra. We also obtain the total deformed Hamiltonian and the energy levels for the system. Finally, we discuss the main reasons to consider two distinct deformation parameters and give our conclusions.

2 Fibonacci Oscillator Algebra

In this section, we consider the two-parameter deformed boson algebra invariant under the quantum group $SU_r(n)$ with $r = q_1/q_2$. For simplicity, we begin with the two-dimensional case. The generalized $SU_{q_1/q_2}(2)$-invariant two-parameter deformed bosonic oscillator algebra is defined by the following deformed commutation relations [10]:

$$\begin{aligned} &a_1 a_2 = \frac{q_1}{q_2} a_2 a_1, \qquad a_1^* a_2^* = \frac{q_2}{q_1} a_2^* a_1^*, \qquad a_1 a_2^* = q_1 q_2 a_2^* a_1, \\ &a_1 a_1^* - q_1^2 a_1^* a_1 = q_2^{2N_B}, \qquad a_2 a_2^* - q_1^2 a_2^* a_2 = a_1 a_1^* - q_2^2 a_1^* a_1, \qquad q_1^{2N_B} = a_2 a_2^* - q_2^2 a_2^* a_2, \end{aligned} \tag{1}$$

where a_i and a_i^*, $i = 1, 2$, are deformed bosonic annihilation and creation operators, respectively. N_B is the total boson number operator and $q_1 \neq q_2$, $(q_1, q_2) \in \mathbf{R}^+$. The deformed number operator for these two-parameter deformed oscillators is

$$a_1^* a_1 + a_2^* a_2 = [N_1 + N_2] = [N_B], \tag{2}$$

whose spectrum is defined by the following Fibonacci basic number $[n]$:

$$[n] = \frac{q_2^{2n} - q_1^{2n}}{q_2^2 - q_1^2}, \tag{3}$$

which is a generalization of the usual q-numbers. Due to this fact, the two-parameter deformed bosonic oscillators in (1) are called Fibonacci oscillators and the deformation parameters q_1 and q_2 are also called as the parameters of the Fibonacci basic integers [10].

We should also emphasize that the Fibonacci oscillator algebra has symmetry under the exchange of the deformation parameters q_1 and q_2. These oscillators give the two-dimensional ordinary bosons in the limit $q_1 = q_2 = 1$. The one-parameter deformed bosonic algebra invariant under the quantum group $SU_{q_1}(2)$ can be obtained in the limit $q_2 = 1$ [68]. Also, the two-dimensional bosonic Newton oscillator algebra invariant under the undeformed group $SU(2)$ can be recovered in the limit $q_1 = q_2 = q$ [12].

One can check that (q_1, q_2)-deformed bosonic algebra in (1) shows $SU_{q_1/q_2}(2)$-symmetry. A $SU_{q_1/q_2}(2)$-matrix can be written in the form

$$T = \begin{pmatrix} a & \frac{q_1}{q_2} b \\ -b^* & a^* \end{pmatrix},$$

where the following commutation relations hold

$$\begin{aligned} &ab = q_1 q_2^{-1} ba, \qquad ab^* = q_1 q_2^{-1} b^* a, \\ &bb^* = b^* b, \qquad a^* a + bb^* = 1, \\ &aa^* + q_1^2 q_2^{-2} b^* b = 1. \end{aligned} \tag{4}$$

By the $SU_{q_1/q_2}(2)$-invariance of the system, it means that the linear transformations

$$\begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = \begin{pmatrix} a & \dfrac{q_1}{q_2} b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad ((a_1^*)' \quad (a_2^*)') = (a_1^* \quad a_2^*) \begin{pmatrix} a^* & -b \\ \dfrac{q_1}{q_2} b^* & a \end{pmatrix}, \tag{5}$$

leads to the same commutation relations as in (1) for $(a_1', (a_1^*)')$ and $(a_2', (a_2^*)')$. We note that the matrix elements of T are assumed to commute with a_1, a_2, a_1^*, a_2^*. We should also emphasize that although the Fibonacci oscillator algebra has two deformation parameters q_1 and q_2, the relations containing the matrix elements of T involve just a single parameter $r = q_1/q_2$.

$SU_{q_1/q_2}(2)$-invariant two-parameter deformed boson algebra in (1) can be generalized to n-dimensional case as follows [10]:

$$\begin{aligned} &a_i a_k = \frac{q_1}{q_2} a_k a_i, \quad i < k, \qquad a_i^* a_k^* = \frac{q_2}{q_1} a_k^* a_i^*, \quad i < k, \\ &a_i a_k^* = q_1 q_2 a_k^* a_i, \quad i \neq k, \qquad a_1 a_1^* - q_1^2 a_1^* a_1 = q_2^{2N_B}, \\ &a_k a_k^* - q_1^2 a_k^* a_k = a_{k-1} a_{k-1}^* - q_2^2 a_{k-1}^* a_{k-1}, \quad k = 2, \ldots, n, \qquad q_1^{2N_B} = a_n a_n^* - q_2^2 a_n^* a_n, \end{aligned} \tag{6}$$

where the total deformed boson number operator for this system is

$$a_1^* a_1 + a_2^* a_2 + \cdots + a_n^* a_n = [N_1 + \cdots + N_n] = [N_B], \tag{7}$$

whose spectrum is given by the Fibonacci basic numbers $[n]$ in (3).

3 Two-Parameter Deformed SUSY Algebra

We now construct the SUSY quantum mechanics for two (q_1, q_2)-deformed bosons with $SU_{q_1/q_2}(2)$-symmetry and two undeformed fermions. The supercharges which are bilinears of these two (q_1, q_2)-deformed bosonic and two undeformed fermionic oscillators are described as follows:

$$\begin{aligned} Q_1 &= a_1^* f_1, \qquad Q_2 = a_2^* f_2, \\ Q_1^* &= f_1^* a_1, \qquad Q_2^* = f_2^* a_2, \end{aligned} \tag{8}$$

where f_i and f_i^*, $i = 1, 2$, are the undeformed fermionic annihilation and creation operators, respectively. These fermionic oscillators satisfy the following anti-commutation relations:

$$\begin{aligned} &\{f_i, f_j^*\} = \delta_{ij}, \qquad \{f_i, f_j\} = 0, \qquad \{f_i^*, f_j^*\} = 0 \\ &f_i^2 = (f_i^*)^2 = 0, \qquad [f_i, M_i] = f_i, \qquad \lfloor f_i^*, M_i \rfloor = -f_i^*, \quad i, j = 1, 2, \end{aligned} \tag{9}$$

where $M_i = f_i^* f_i$ is the fermionic number operator. The odd generators called supercharges in (8) are nilpotent:

$$Q_1^2 = Q_2^2 = (Q_1^*)^2 = (Q_2^*)^2 = 0, \tag{10}$$

which can be obtained from (9). We now look for some particular relations between the deformed bosonic a_i and undeformed fermionic operators f_m as follows:

$$a_i f_m = r f_m a_i, \quad a_i f_m^* = s f_m^* a_i, \quad i, m = 1, 2, \tag{11}$$

where r and s are some parameters which can be determined from (1) and (9) as $r = s = 1$. It means that the bosonic and fermionic operators in our construction commute with each other.

With the above considerations in mind, we have the following (q_1, q_2)-deformed SUSY algebra for two (q_1, q_2)-deformed bosons and two undeformed fermions:

$$\{Q_1, Q_2\}_{(q_1^2+q_2^2)/2q_1q_2} = 0, \qquad \{Q_1, Q_2^*\}_{q_1^{-1}q_2^{-1}} = 0, \tag{12}$$

$$\{Q_1, Q_1^*\}_{2/(q_1^2+q_2^2)} = H_1 = a_1^* a_1 + \left(\frac{q_1^{2N_B} + q_2^{2N_B}}{q_1^2 + q_2^2}\right) f_1^* f_1, \tag{13}$$

$$\{Q_2, Q_2^*\}_{2/(q_1^2+q_2^2)} = H_2 = a_2^* a_2 + \left(\frac{q_1^{2N_B} + q_2^{2N_B}}{q_1^2 + q_2^2}\right) f_2^* f_2, \tag{14}$$

$$[H_1, Q_1]_{(q_1^2+q_2^2)/2} = 0, \qquad [H_2, Q_2]_{(q_1^2+q_2^2)/2} = 0, \tag{15}$$

$$[H_1, Q_2]_{(q_1^2+q_2^2)/2} = 0, \qquad [H_2, Q_1]_{(q_1^2+q_2^2)/2} = 0, \tag{16}$$

where $\{A, B\}_x = AB + xBA$ and $[A, B]_x = AB - xBA$, and also $N_B = N_1 + N_2$. One can show that this (q_1, q_2)-deformed SUSY algebra is invariant under the $SU_{q_1/q_2}(2)$-transformation of (q_1,q_2)-deformed bosonic oscillators. Moreover, the total deformed Hamiltonian for the two-dimensional system becomes

$$H = H_1 + H_2 = (a_1^* a_1 + a_2^* a_2) + \left(\frac{q_1^{2N_B} + q_2^{2N_B}}{q_1^2 + q_2^2}\right)(f_1^* f_1 + f_2^* f_2), \tag{17}$$

where $N_B = N_1 + N_2$. Therefore, this Hamiltonian describes a two-parameter deformation of the Hamiltonian for the supersymetric oscillator in quantum mechanics [32]. In the limit

$q_2 = q_1 = 1$, this Hamiltonian goes to the standard SUSY Hamiltonian for the system containing two undeformed bosons and two undeformed fermions.

We can extend the above (q_1, q_2)-deformed SUSY algebra to the system containing n deformed bosons and n undeformed fermions. We then have the following $2n$ supercharges:

$$Q_i = a_i^* f_i, \qquad Q_i^* = f_i^* a_i. \tag{18}$$

Thus, the generalized (q_1, q_2)-deformed SUSY algebra for $SU_{q_1/q_2}(n)$-invariant (q_1, q_2)-deformed bosons and undeformed fermions is

$$\{Q_i, Q_j\}_{(q_1^2+q_2^2)/2q_1q_2} = 0, \quad i < j, \tag{19}$$

$$\{Q_i, Q_j^*\}_{q_1^{-1}q_2^{-1}} = 0, \quad i \neq j, \tag{20}$$

$$[H_i, Q_j]_{(q_1^2+q_2^2)/2} = 0, \quad i \geq j, \tag{21}$$

$$[H_i, Q_j]_{(q_1^2+q_2^2)/2} = 0, \quad i < j, \tag{22}$$

$$\{Q_i, Q_i^*\}_{2/(q_1^2+q_2^2)} = H_i = a_i^* a_i + \left(\frac{q_1^{2N_B} + q_2^{2N_B}}{q_1^2 + q_2^2} \right) f_i^* f_i, \tag{23}$$

where N_B is the total boson number operator. We note that from (21), the Hamiltonian H_i in (23) does not commute with the supercharges Q_i unless $q_1 = q_2 = 1$. Therefore, the supercharges Q_i are not conserved and the Hamiltonian H_i does not remain invariant under the (q_1, q_2)-deformed SUSY algebra constructed. Moreover, the total deformed Hamiltonian for the n-dimensional system becomes

$$H = \sum_{i=1}^{n} H_i = \sum_{i=1}^{n} a_i^* a_i + \left(\frac{q_1^{2N_B} + q_2^{2N_B}}{q_1^2 + q_2^2} \right) \sum_{i=1}^{n} f_i^* f_i, \tag{24}$$

where $N_B = N_1 + N_2 + \cdots + N_n$. The Fock space representation of the above (q_1, q_2)-deformed SUSY algebra will be discussed in the next section.

4 Fock Space Representation of the (q_1, q_2)-Deformed SUSY Algebra

In this section, we discuss the Fock space representation of the (q_1, q_2)-deformed SUSY algebra in equations (12–17). For the bosonic sector, let $|n_1, n_2\rangle$ be the Fock space basis and the ground state satisfies $a_i|0, 0\rangle = 0$, $i = 1, 2$. From (1), the representations of the operators a_1, a_2, a_1^*, a_2^* are as follows:

$$\begin{aligned} a_1|n_1, n_2\rangle &= q_2^{n_2}\sqrt{[n_1]}|n_1 - 1, n_2\rangle, & a_1^*|n_1, n_2\rangle &= q_2^{n_2}\sqrt{[n_1+1]}|n_1 + 1, n_2\rangle, \\ a_2|n_1, n_2\rangle &= q_1^{n_1}\sqrt{[n_2]}|n_1, n_2 - 1\rangle, & a_2^*|n_1, n_2\rangle &= q_1^{n_1}\sqrt{[n_2+1]}|n_1, n_2 + 1\rangle, \\ a_1^* a_1|n_1, n_2\rangle &= q_2^{2n_2}[n_1]|n_1, n_2\rangle, & a_2^* a_2|n_1, n_2\rangle &= q_1^{2n_1}[n_2]|n_1, n_2\rangle, \end{aligned} \tag{25}$$

where $n_1, n_2 = 0, 1, 2, \ldots,$ and $[n_i]$ is defined by (3). The orthonormal (q_1, q_2)-deformed boson state $|n_1, n_2\rangle$ is obtained by applying the Fibonacci oscillator creation operators (a_i^*) on the ground state successively as

$$|n_1, n_2\rangle = \frac{1}{\sqrt{[n_1]![n_2]!}} (a_2^*)^{n_2} (a_1^*)^{n_1} |0, 0\rangle. \tag{26}$$

From the (q_1, q_2)-deformed bosonic oscillator algebra in (6), we also have the following representations:

$$a_i|n_1, n_2, \ldots, n_i, \ldots, n_n\rangle = q_1^{\sum_{k=1}^{i-1} n_k}(\sqrt{[n_i]})q_2^{\sum_{k=i+1}^{n} n_k}|n_1, n_2, \ldots, n_i - 1, \ldots, n_n\rangle, \quad (27)$$

$$a_i^*|n_1, n_2, \ldots, n_i, \ldots, n_n\rangle = q_1^{\sum_{k=1}^{i-1} n_k}(\sqrt{[n_i + 1]})q_2^{\sum_{k=i+1}^{n} n_k}|n_1, n_2, \ldots, n_i + 1, \ldots, n_n\rangle, \quad (28)$$

where $a_i^* a_i = (q_1^2)^{\sum_{k=1}^{i-1} n_k}[n_i](q_2^2)^{\sum_{k=i+1}^{n} n_k}$, and $[n_i]$ is defined by (3).

For the fermionic sector, we have two undeformed fermion operators satisfying (9). Therefore, there exist four types of quantum states for this sector as $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$. Thus, the following representations for the operators f_i, f_i^*, $i = 1, 2$, can be written:

$$\begin{array}{llll}
f_1|\uparrow\uparrow\rangle = |\downarrow\uparrow\rangle, & f_2|\uparrow\uparrow\rangle = |\uparrow\downarrow\rangle, & f_1^*|\uparrow\uparrow\rangle = 0, & f_2^*|\uparrow\uparrow\rangle = 0, \\
f_1|\downarrow\downarrow\rangle = 0, & f_2|\downarrow\downarrow\rangle = 0, & f_1^*|\downarrow\downarrow\rangle = |\uparrow\downarrow\rangle, & f_2^*|\downarrow\uparrow\rangle = 0, \\
f_1|\uparrow\downarrow\rangle = |\downarrow\downarrow\rangle, & f_2|\uparrow\downarrow\rangle = 0, & f_1^*|\uparrow\downarrow\rangle = 0, & f_2^*|\uparrow\downarrow\rangle = |\uparrow\uparrow\rangle, \\
f_1|\downarrow\uparrow\rangle = 0, & f_2|\downarrow\uparrow\rangle = |\downarrow\downarrow\rangle, & f_1^*|\downarrow\uparrow\rangle = |\uparrow\uparrow\rangle, & f_2^*|\downarrow\downarrow\rangle = |\downarrow\uparrow\rangle,
\end{array} \quad (29)$$

which show that the state $|\downarrow\downarrow\rangle$ is the ground state of the fermionic sector. For the fermionic number operators M_i, $i = 1, 2$, we have the following representations:

$$\begin{array}{llll}
M_1|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle, & M_2|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle, & M_1|\downarrow\uparrow\rangle = 0, & M_2|\downarrow\uparrow\rangle = |\downarrow\uparrow\rangle, \\
M_1|\uparrow\downarrow\rangle = |\uparrow\downarrow\rangle, & M_2|\uparrow\downarrow\rangle = 0, & M_1|\downarrow\downarrow\rangle = 0, & M_2|\downarrow\downarrow\rangle = 0.
\end{array} \quad (30)$$

With the above results in mind, it turns out that there are four types of states for the supercharges and the Hamiltonian in the full Hilbert space as $|(n_1, n_2); \uparrow\uparrow\rangle$, $|(n_1, n_2); \uparrow\downarrow\rangle$, $|(n_1, n_2); \downarrow\uparrow\rangle$, $|(n_1, n_2); \downarrow\downarrow\rangle$. Acting on them with the supercharges yields

$$\begin{aligned}
&Q_1|(n_1, n_2); \uparrow\uparrow\rangle = q_2^{n_2}\sqrt{[n_1 + 1]}\,|(n_1 + 1, n_2); \downarrow\uparrow\rangle, \\
&Q_1|(n_1, n_2); \downarrow\downarrow\rangle = 0, \qquad Q_1|(n_1, n_2); \downarrow\uparrow\rangle = 0, \\
&Q_1|(n_1, n_2); \uparrow\downarrow\rangle = q_2^{n_2}\sqrt{[n_1 + 1]}|(n_1 + 1, n_2); \downarrow\downarrow\rangle,
\end{aligned} \quad (31)$$

$$\begin{aligned}
&Q_2|(n_1, n_2); \uparrow\uparrow\rangle = q_1^{n_1}\sqrt{[n_2 + 1]}|(n_1, n_2 + 1); \uparrow\downarrow\rangle, \\
&Q_2|(n_1, n_2); \downarrow\downarrow\rangle = 0, \qquad Q_2|(n_1, n_2); \uparrow\downarrow\rangle = 0, \\
&Q_2|(n_1, n_2); \downarrow\uparrow\rangle = q_1^{n_1}\sqrt{[n_2 + 1]}|(n_1, n_2 + 1); \downarrow\downarrow\rangle,
\end{aligned} \quad (32)$$

$$\begin{aligned}
&Q_1^*|(n_1, n_2); \uparrow\uparrow\rangle = 0, \qquad Q_1^*|(n_1, n_2); \uparrow\downarrow\rangle = 0, \\
&Q_1^*|(n_1, n_2); \downarrow\downarrow\rangle = q_2^{n_2}\sqrt{[n_1]}|(n_1 - 1, n_2); \uparrow\downarrow\rangle, \\
&Q_1^*|(n_1, n_2); \downarrow\uparrow\rangle = q_2^{n_2}\sqrt{[n_1]}|(n_1 - 1, n_2); \uparrow\uparrow\rangle,
\end{aligned} \quad (33)$$

$$\begin{aligned}
&Q_2^*|(n_1, n_2); \uparrow\uparrow\rangle = 0, \qquad Q_2^*|(n_1, n_2); \downarrow\uparrow\rangle = 0, \\
&Q_2^*|(n_1, n_2); \downarrow\downarrow\rangle = q_1^{n_1}\sqrt{[n_2]}|(n_1, n_2 - 1); \downarrow\uparrow\rangle, \\
&Q_2^*|(n_1, n_2); \uparrow\downarrow\rangle = q_1^{n_1}\sqrt{[n_2]}|(n_1, n_2 - 1); \uparrow\uparrow\rangle.
\end{aligned} \quad (34)$$

Similarly, applying the total deformed Hamiltonian in (17) on these states yields

$$H|(n_1,n_2);\uparrow\uparrow\rangle = \left\{[n_1+n_2]+2\left(\frac{q_1^{2(n_1+n_2)}+q_2^{2(n_1+n_2)}}{q_1^2+q_2^2}\right)\right\}|(n_1,n_2);\uparrow\uparrow\rangle, \quad (35)$$

$$H|(n_1,n_2);\uparrow\downarrow\rangle = \left\{[n_1+n_2]+\left(\frac{q_1^{2(n_1+n_2)}+q_2^{2(n_1+n_2)}}{q_1^2+q_2^2}\right)\right\}|(n_1,n_2);\uparrow\downarrow\rangle, \quad (36)$$

$$H|(n_1,n_2);\downarrow\uparrow\rangle = \left\{[n_1+n_2]+\left(\frac{q_1^{2(n_1+n_2)}+q_2^{2(n_1+n_2)}}{q_1^2+q_2^2}\right)\right\}|(n_1,n_2);\downarrow\uparrow\rangle, \quad (37)$$

$$H|(n_1,n_2);\downarrow\downarrow\rangle = [n_1+n_2]|(n_1,n_2);\downarrow\downarrow\rangle, \quad (38)$$

where $[n_i]$ is the Fibonacci basic number in (3). From these representations, the first few excited states and their degeneracies can be obtained as in the following scheme:

The energy eigenvalues	*The energy eigenstates*	
$E=0$	$\|(0,0);\downarrow\downarrow\rangle$ $\Rightarrow$	The ground state
$E=1$	$\|(0,1);\downarrow\downarrow\rangle$,	$\|(1,0);\downarrow\downarrow\rangle$
$E=2$	$\|(0,1);\uparrow\downarrow\rangle$,	$\|(1,0);\uparrow\downarrow\rangle$
	$\|(0,1);\downarrow\uparrow\rangle$,	$\|(1,0);\downarrow\uparrow\rangle$
$E=3$	$\|(1,0);\uparrow\uparrow\rangle$,	$\|(0,1);\uparrow\uparrow\rangle$

In contrast to these levels, the energy eigenvalues corresponding to all other excited states depend on the deformation parameters q_1 and q_2.

5 Discussion and Conclusions

In this paper, we studied the system of n (q_1,q_2)-deformed bosons with $SU_{q_1/q_2}(n)$-symmetry and n ordinary fermions. We explicitly discussed the system containing two (q_1,q_2)-deformed bosons and two undeformed fermions. We made use of these results to construct a two-parameter deformed SUSY algebra by realizing SUSY generators as bilinears of n (q_1,q_2)-deformed bosons with $SU_{q_1/q_2}(n)$-symmetry and n ordinary fermions. We also discussed the Fock space representation of the algebra constructed and found the energy eigenvalues of the total deformed Hamiltonian for the two-dimensional system. The remarkable differences between our construction (19–24) and the earlier q-deformed SUSY algebra constructions [4, 19, 20, 23, 26, 27, 38, 47, 49, 50, 56, 65, 71] are as follows:

(i) We use the (q_1,q_2)-deformed bosonic oscillators called Fibonacci oscillators [10] which are invariant under the quantum group $SU_r(n)$ with $r=q_1/q_2$.

(ii) In our construction, the bosonic and fermionic sectors operators commute with each other. In this sense, (q_1,q_2)-deformed bosons and undeformed fermions are independent in our model.

(iii) In our model, the Hamiltonian H_i in (23) does not commute with the supercharges Q_i unless $q_1=q_2=1$. So, the supercharges are not conserved and the Hamiltonian H_i does not remain invariant under the (q_1,q_2)-deformed SUSY algebra. However, in our construction the total deformed Hamiltonian H in (24) coincides with the total bosonic and fermionic number operators in the limit $q_1 = q_2 = 1$.

We now would like to mention some physical applications of the Fibonacci oscillators which are invariant under the quantum group $SU_r(n)$ with $r = q_1/q_2$ as follows:

(i) The two-parameter deformed quantum group invariant bosonic oscillators in (6) and (7) have recently revealed many interesting results in the framework of statistical thermodynamics [5, 7, 13]. For instance, a two-parameter deformed quantum group boson gas with $SU_r(2)$-symmetry where $r = q_1/q_2$ shows the Bose-Einstein condensation for low temperatures in the interval $q_2 > q_1 > 0$ [7] whereas it behaves as a fermionic gas at the value of $(q_1^2 + q_2^2) \approx 4.16$ for high temperatures. Therefore, for high temperatures, the deformation parameters q_1 and q_2 of the $SU_{q_1/q_2}(2)$-invariant boson model serve as interpolating objects between bosonic and fermion-like behaviours of the system [5]. In such a model, the limit $q_1 = q_2 = 1$ gives the free boson gas results.

(ii) One or multiparameter deformed versions of quantum (super)algebras have produced many new results in the framework of integrable models and solvable lattice models. For instance, it was recently shown [62] that the quantum algebraic structure of the spin 1/2 XXZ chain with twisted periodic boundary conditions is a two-parameter deformed algebra $SU_{q,t}(2)$. Another example is that the work of Reshetikhin [70] on multiparametric quantum algebras was used to construct integrable multiparametric quantum spin chains [39, 40]. Hence, possible applications of the Fibonacci oscillator algebra with $SU_{q_1/q_2}(n)$-symmetry in the framework of integrable models with twisted boundary conditions would be another direction of this study which would provide new results for investigations related with correlated electron systems.

We should remark that the above interesting applications give the main motivation and the importance for studies on two-parameter deformed quantum group invariant bosonic oscillators.

Some important limiting cases of the present (q_1, q_2)-deformed SUSY algebra should be mentioned: In the limit $q_1 = q_2 = 1$, the conventional $N = 2$ SUSY algebra can be recovered [76]. The limit $q_1 = q_2 = q$ gives an alternative example of the q-deformed $N = 2$ SUSY algebra constructed from the q-deformed bosonic and fermionic Newton oscillators [4]. When we take the limit $q_2 = 1$, we find another one-parameter deformed SUSY algebra constructed from $SU_{q_1}(n)$-invariant bosons and undeformed fermions.

Beside the present application of the Fibonacci oscillators to the SUSY quantum mechanics, we would also like to discuss the main reasons to consider two distinct deformation parameters (q_1, q_2). These reasons are two-fold. The first one is related to the following quantum algebraical properties:

(i) The Fibonacci oscillator discussed in Sect. 2 offers a unification of oscillators related to quantum groups [10].

(ii) The Fibonacci oscillators are the most general oscillators having the property of spectrum degeneracy. In this sense, the eigenstates $|n\rangle$ of $[N_B]$ in (7) have the same bosonic degeneracy as the ordinary n-dimensional quantum harmonic oscillator.

(iii) The Fibonacci oscillator algebra is the most general quantum group invariant bosonic oscillator algebra. The invariance quantum group of this oscillator is the $SU_r(n)$ with $r = q_1/q_2$. In this sense, if the quantum group symmetry is preserved, then the number of deformation parameters in n dimensions should be just two.

(iv) Although there are some studies showing a close connection between relativity, one-parameter deformed bosonic oscillators called q-oscillators and difference operators [11, 60, 69], the multidimensional Fibonacci oscillator corresponding to the basic number definition in (3) can be interpreted as a relativistic oscillator corresponding to the bound state of two bosonic particles with masses m_1 and m_2 [10]. The deformation parameters q_1 and q_2 are then related to the masses by

$$q_1^2 = 1 + \frac{2\omega}{m_1}, \qquad q_2^2 = 1 + \frac{2\omega}{m_2}, \tag{39}$$

where ω is the oscillator frequency. The energy spectrum is given by

$$E_n = \sqrt{m_1^2 + K_n^2} + \sqrt{m_2^2 + K_n^2}, \tag{40}$$

$$K_n^2 = \left(\frac{m_1 m_2}{m_1 + m_2}\right)^2 \left(\frac{q_1^2 + q_2^2 - 2}{q_1^2 - q_2^2}\right)\left(\left(\frac{1+q_1^2}{2}\right)^d q_1^{2n} - \left(\frac{1+q_2^2}{2}\right)^d q_2^{2n}\right), \tag{41}$$

where d represents the dimension of the system [10]. Therefore, the additional parameter q_2 has a physical significance so that it can be related to the mass of the second bosonic particle in a two-particle relativistic quantum harmonic oscillator bound state.

On the other hand, the second reason comes originally from the following phenomenological studies:

(i) The quantum algebra with two deformation parameters may have more flexibility when dealing with application to the concrete physical models [46]. Although any quantum algebra with one or more deformation parameters may be mapped onto the standard single-parameter case [28, 67], the physical results obtained from a (p,q)-deformed oscillator system are not the same. In particular, it is recently argued that the (p,q)-deformed phonon systems may be useful and effective in order to deal with anharmonicity or/and interactions of phonons in the realistic condensed matter [29, 43–45].

(ii) The recent results arising from the application of the qp-rotator model with $U_{qp}(u_2)$-symmetry to the description of rotational bands of various atomic nuclei show also the importance for the introducing second deformation parameter. In this sense, the results obtained from the qp-rotator model are better than the ones derived from the q-rotator model and the κ-Poincare model [14, 18, 55]. In such an application, the two deformation parameters of the qp-rotator model had an interpretation as inertial parameters that describe the softness of the deformed and superdeformed nuclei.

(iii) Recently, quantum algebra with one deformation parameter has been used in phenomenological description of particle properties [41]. In particular, q-bosons have been used to describe unusual behaviour of the intercept (or the strength) λ of the two-particle correlation function. This technique gives a direct correspondence between the deformation parameter q and the intercept λ [8]. So, what about the higher order correlations? The answer to this question gives the importance of theories with two deformation parameters to some extent. The two-parameter deformed oscillator is a candidate to describe an explicit form of the intercepts $\lambda^{(n)}$ of n-particle correlation functions with $n \geq 3$ of identical pions or kaons [1]. However, we should note that there is a remarkable difference between the Fibonacci oscillators given in (6) and the so called q-bosons. Strictly speaking, the q-bosons [16, 58, 64] do not have a covariance under the action of the quantum group $SU_q(n)$.

All above considerations give the reasons to consider two distinct deformation parameters, and therefore show the importance and requirement to think Fibonacci oscillators in physical applications.

To conclude, some open problems parallel to this study are as follows: It will be interesting to investigate a fractional SUSY structure when the deformation parameter $r = q_1/q_2$ is a root of unity which would hopefully provide some new statistical insights into studies on fractional statistics.

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